ISSN 2229-5518

Generalised Gaussian Quadrature over a Sphere

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Abstract — This paper presents a Generalised Gaussian quadrature method for the evaluation of volume integral $I = \iiint_v f(x,y,z)dx\,dy\,dz$, where $f(x,y,z)$ is arbitrary function and v refers to the volume of spherical region bounded by $\{(x,y,z)/-a \le x \le a, -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}, -\sqrt{a^2-x^2-y^2} \le z \le \sqrt{a^2-x^2-y^2}\}$, volume integral is convert to surface integral by using Gauss divergence theorem then we have applied the Generalised Gaussian quadrature rules over a circle region to evaluate the typical volume integrals over the spherical region with various values of a. The efficacy of this method is finally shown by numerical examples.

Index Terms— Finite element method , Generalised Gaussian Quadrature , spherical region.

1. Introduction

THE finite element method has proven to be an efficient tool for the numerical analysis of two- or three-dimensional structures of whatever complexity, in mechanical, thermal or other physical problems. It is widely recognized that computational cost increases greatly with structure complexity, being larger with three-dimensional analyses than with two-dimensional ones. It is therefore desirable to devise simplified approaches that may provide a reduction in overall computational effort. An example of considerable importance is the study of bodies of revolution where a three dimensional problem is solved by a two-dimensional analysis. In particular, they are used for Problems involving calculations Volume, center of mass, moment of inertia and other geometric properties of rigid homogeneous solids frequently arise in a large number of engineering applications, in CAD/CAE/CAM applications in geometric modeling as well as in robotics and similar problems in other areas of engineering which are very difficult to analyze using analytical techniques, These problems can be solved using the finite element method.

A good overview of various method for evaluating volume integrals is given by Lee and Requicha [8] evaluation of volume integrals by transforming the volume integral to a surface integral and then into a parametric line integral is given by Timmer and Stern [4], Cattani and Paoluzzi [2] gave a symbolic solution to both volume and surface integration of polynomials by using a triangulation of solid boundary, Nagaraja and Rathod [9] have discussed the volume integral of a function is express to sum of four integrals over the unit triangle by using gauss divergence theorem, shivaram [11] evaluation of surface integral of arbitrary function over circle region by using generalized Gaussian quadrature rule.

The paper is organized as follows. In Section II volume of the sphere is equal to 8 times the volume in the first octant. In Section III we will introduce the Generalized Gaussian quadrature formula over a circle region of various values a. and In Section IV we compare the numerical results with some illustrative examples.

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ISSN 2229-5518

2 Generalised Gaussian Quadrature over a Spherical Region

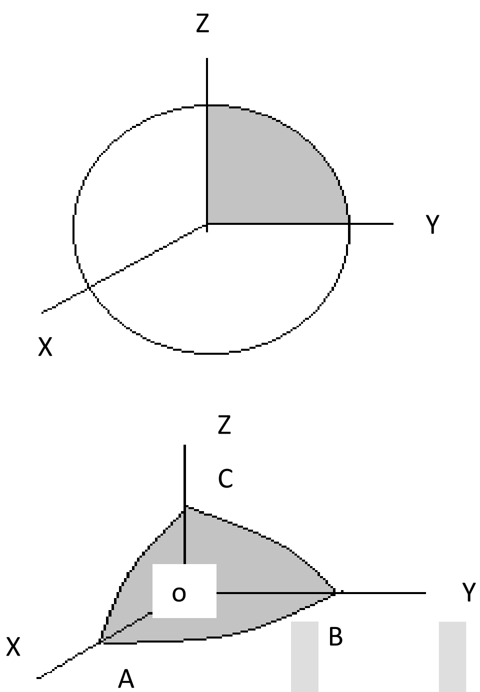

Fig. 1 a) Volume of the Spherical region

b) OABC is piecewise smooth and is comprised of four surfaces

The Numerical integration of an arbitrary function $f(x,y,z)$ over a Spherical region is given by

$$I = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f(x,y,z)dz\,dy\,dx$$

$$= 8\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} \int_{0}^{\sqrt{a^2-x^2-y^2}} f(x,y,z)dz\,dy\,dx$$

Generalised Gaussian quadrature rule for integrating of volume integral bounded by spherical region V = $\{(x,y,z)/0 \le x \le a, 0 \le y \le \sqrt{a^2-x^2}, 0 \le z \le \sqrt{a^2-x^2-y^2}\}$ with $a = 0.5,\ 1,\ 3$ and these volume integral convert to surface integral using Gauss divergence theorem.

3 Formulation of Integrals over a Quarter- Circle Region

The Numerical integration of an arbitrary function f over a quarter circle is given by

$$\mathrm{I} = \iint_C f(x,y)dx\,dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)dy\,dx \qquad (1)$$

Where a is the radius of the circle

The integral of the eqn.(1) can be transformed to the square $\{(\xi,\eta)/\ 0 \le \xi \le 1, 0 \le \eta \le 1\}$. Transformation is

$$x = a\,\xi \quad and \quad y = a\,\eta\,\sqrt{1-\xi^2} \qquad (2)$$

We have

$$\mathrm{I} = \int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)dy\,dx$$

$$= \int_0^1 \int_0^1 f\big(x(\xi,\eta), y(\xi,\eta)\big)\,J\,d\xi\,d\eta \qquad (3)$$

Where $J(\xi,\eta)$ is the Jacobians of the transformation

$$J(\xi,\eta) = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} = a^2\sqrt{1-\xi^2}$$

From eqn.(3) , we can write as

$$I = \int_0^1 \int_0^1 f\left(a\,\xi, a\,\eta\,\sqrt{1-\xi^2}\right) a^2\sqrt{1-\xi^2}\,d\xi\,d\eta$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a^2\sqrt{1-\xi^2}\ w_i w_j\, f\big(x(\xi_i,\eta_j), y(\xi_i,\eta_j)\big) \quad (4)$$

Where ξ_i, η_j are Gaussian points and w_i, w_j are corresponding weights. We can rewrite eqn. (4) as

$$\mathrm{I} = \sum_{k}^{N=n\times n} W_k f(x_k, y_k) \qquad (5)$$

$$\text{Where} \quad W_k = a^2\sqrt{1-\xi^2}\ w_i w_j\ , \qquad (5a)$$

$$x_k = a\,\xi\ , \qquad (5b)$$

$$y_k = a\,\eta\,\sqrt{1-\xi^2}\ , \qquad (5c)$$

if $i,j,k = 1,2,3,----$,

ISSN 2229-5518

we find out new Gaussian points(x_k , y_k) and weights coefficients W_k of various order N =5,10,15,20 by using eqn. (5a), (5b) and (5c) and Tabulated in Table 1 and 2

TABLE 1

Gaussian points and weights coefficient over the region with a=1 and N = 5

x_k	y_k	W_k
0.0056522282	0.0056521379	0.0004429668
0.0734303717	0.0056369691	0.0027435258
0.2849574044	0.0054178878	0.0058445536
0.6194822640	0.0044370593	0.0057863677
0.9157580830	0.0022706706	0.0017614267
0.0056522282	0.0734291987	0.0027509084
0.0734303717	0.0732321351	0.0170378176
0.2849574044	0.0703859617	0.0362957910
0.6194822640	0.0576436240	0.0359344457
0.9157580830	0.0294991960	0.0109387957
0.0056522282	0.2849528525	0.0060972512
0.0734303717	0.2841881181	0.0377634717
0.2849574044	0.2731431219	0.0804478077
0.6194822640	0.2236945980	0.0796469039
0.9157580830	0.1144759879	0.0242452942
0.0056522282	0.6194723684	0.0073709513
0.0734303717	0.6178098764	0.0456521641
0.2849574044	0.5937986412	0.0972531486
0.6194822640	0.4863001763	0.0962849380
0.9157580830	0.2488647182	0.0293100741
0.0056522282	0.9157434547	0.0043845316
0.0734303717	0.9132858532	0.0271557016
0.2849574044	0.8777909181	0.0578499953
0.6194822640	0.7188798502	0.0572740656
0.9157580830	0.3678876548	0.0174347841

TABLE 2

Gaussian points and weights coefficient over the region with a=0.5 and N = 5

x_k	y_k	W_k
0.002826114	0.002826069	0.000110742
0.036715186	0.002818485	0.000685881
0.142478702	0.002708944	0.001461138
0.309741132	0.00221853	0.001446592
0.457879042	0.001135335	0.000440357
0.002826114	0.036714599	0.000687727
0.036715186	0.036616068	0.004259454
0.142478702	0.035192981	0.009073948
0.309741132	0.028821812	0.008983611
0.457879042	0.014749598	0.002734699
0.002826114	0.142476426	0.001524313
0.036715186	0.142094059	0.009440868
0.142478702	0.136571561	0.020111952
0.309741132	0.111847299	0.019911726
0.457879042	0.057237994	0.006061324
0.002826114	0.309736184	0.001842738
0.036715186	0.308904938	0.011413041
0.142478702	0.296899321	0.024313287
0.309741132	0.243150088	0.024071235
0.457879042	0.124432359	0.007327519
0.002826114	0.457871727	0.001096133
0.036715186	0.456642927	0.006788925
0.142478702	0.438895459	0.014462499
0.309741132	0.359439925	0.014318516
0.457879042	0.183943827	0.004358696

4 Numerical Result

Exact value	N	Computed value
1) $\int_0^{0.5}\int_0^{\sqrt{0.5^2-x^2}}\int_0^{\sqrt{0.5^2-x^2-y^2}} xyz\,dzdydx = \int_0^{0.5}\int_0^{\sqrt{0.5^2-x^2}} \frac{1}{2}xy(\frac{1}{4}-x^2-y^2)dydx$ = 0.0003255208333	5	0.000325512
	10	0.000328739
	15	0.000325477
	20	0.000325502
2) $\int_0^{0.5}\int_0^{\sqrt{0.5^2-x^2}}\int_0^{\sqrt{0.5^2-x^2-y^2}} \sqrt{x+y+z}\,dzdydx = \int_0^{0.5}\int_0^{\sqrt{0.5^2-x^2}} [\frac{2}{3}(\frac{1}{2}\sqrt{1-4x^2-4y^2}+x+y)^{\frac{3}{2}}-\frac{2}{3}(x+y)^{\frac{3}{2}}]dydx$ = 0.04851863601	5	0.048666861
	10	0.048305638
	15	0.048519698
	20	0.048518793
3) $\int_0^{1}\int_0^{\sqrt{1^2-x^2}}\int_0^{\sqrt{1^2-x^2-y^2}} [\frac{x^2y+z}{x+y}]dzdydx$	5	0.449923809
	10	0.446724537
	15	0.448256687

$= \int_0^1 \int_0^{\sqrt{1^2-x^2}} [\frac{1-x^2-y^2}{2(x+y)} + \frac{x^2 y\sqrt{1-x^2-y^2}}{(x+y)}]dydx$ = 0.4488168268	20	0.448827578
4) $\int_0^1 \int_0^{\sqrt{1^2-x^2}} \int_0^{\sqrt{1^2-x^2-y^2}} \frac{xyz}{\sqrt{x^2+y^2+z^2}} dzdydx$ $= \int_0^1 \int_0^{\sqrt{1^2-x^2}} [xy - xy\sqrt{x^2+y^2}]dydx$ = 0.02500000000	5 10 15 20	0.02500014 0.02513677 0.02499524 0.02500001
5) $\int_0^3 \int_0^{\sqrt{3^2-x^2}} \int_0^{\sqrt{3^2-x^2-y^2}} \frac{xyz}{\sqrt{x^2+y^2+z^2}} dzdydx$ $= \int_0^3 \int_0^{\sqrt{3^2-x^2}} [3xy - xy\sqrt{x^2+y^2}]dydx$ = 6.075000000	5 10 15 20	6.07503109 6.08239570 6.07504415 6.07499423
6) $\int_0^3 \int_0^{\sqrt{3^2-x^2}} \int_0^{\sqrt{3^2-x^2-y^2}} (x+y+z)dzdydx$ $= \int_0^3 \int_0^{\sqrt{3^2-x^2}} [x\sqrt{9-x^2-y^2} + y\sqrt{9-x^2-y^2} \frac{-1}{2}(x^2+y^2) + \frac{9}{2}]dydx$ = 47.71293843	5 10 15 20	47.8608227 47.6743991 47.7121540 47.7120449

5 CONCLUSIONS

In this paper Volume integral(triple) of arbitrary function over a spherical region $\{(x,y,z)/-a \le x \le a, -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}, -\sqrt{a^2-x^2-y^2} \le z \le \sqrt{a^2-x^2-y^2}\}$ with a = 0.5, 1, 2 convert to double integral by using Gauss divergence theorem. We have applied Generalised Gaussian quadrature rule to evaluate the typical integrals The results obtained are in excellent agreement with the exact value.

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